1. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 = \frac{1}{4}.$$

Prove that

$$\frac{1}{\sqrt{b^2+c^2}} + \frac{1}{\sqrt{c^2+a^2}} + \frac{1}{\sqrt{a^2+b^2}} \le \frac{\sqrt{2}}{(a+b)(b+c)(c+a)}.$$

North Macedonia

Solution. Using AM-QM and AM-GM inequalities, we have

$$\begin{aligned}
\frac{1}{\sqrt{b^2+c^2}} + \frac{1}{\sqrt{c^2+a^2}} + \frac{1}{\sqrt{a^2+b^2}} &\le \frac{\sqrt{2}}{b+c} + \frac{\sqrt{2}}{c+a} + \frac{\sqrt{2}}{a+b} \\
&= \sqrt{2} \cdot \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\
&\le \sqrt{2} \cdot \frac{a^2+b^2+c^2+3(\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2})}{(a+b)(b+c)(c+a)} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{8(a^2+b^2+c^2)}{(a+b)(b+c)(c+a)} \\
&= \frac{\sqrt{2}}{(a+b)(b+c)(c+a)}.
\end{aligned}$$

2. Let ABC be a triangle such that $AB < AC$. Let the excircle opposite to A be tangent to the lines AB, AC and BC at points D, E and F, respectively, and let J be its centre. Let P be a point on the side BC. The circumcircles of the triangles BDP and CEP intersect for the second time at Q. Let R be the foot of the perpendicular from A to the line FJ. Prove that the points P, Q and R are collinear.

(The *excircle* of a triangle ABC opposite to A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Bulgaria

Solution. Since the quadrilateral $BDQP$ is cyclic, we have

$$\angle DQP = 180^\circ - \angle DBP = \angle ABC.$$

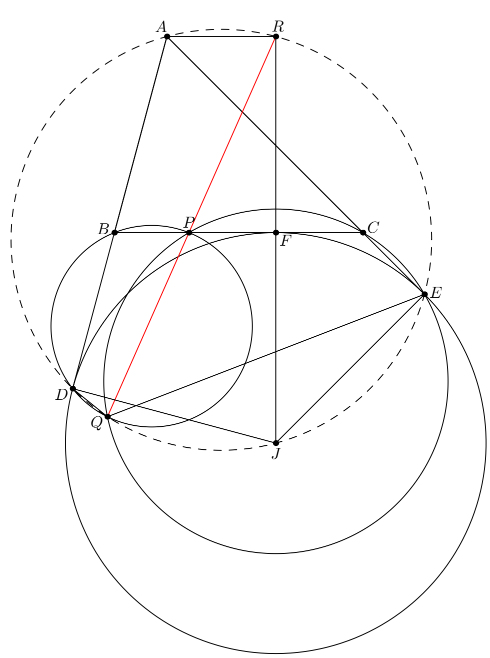
Analogously from the cyclic quadrilateral $CEQP$, we obtain $\angle EQP = \angle ACB$. Hence

$$\angle DQE = \angle DQP + \angle EQP = \angle ABC + \angle ACB = 180^\circ - \angle BAC = 180^\circ - \angle DAE,$$

so Q lies on the circumcircle of $\triangle ADE$. On the other hand, since

$$\angle ADJ = \angle AEJ = \angle ARJ = 90^\circ,$$

we can conclude that R and J also lie on the circumcircle of $\triangle ADE$.



Note that the quadrilateral $BDJF$ and $DJQR$ are cyclic, so we have

$$\angle DQR = \angle DJR = \angle DJF = 180^\circ - \angle DBF = \angle ABC.$$

Then, using $\angle DQP = \angle ABC$, we obtain $\angle DQR = \angle DQP$, so the result follows.

3. Find all triples of positive integers (x, y, z) that satisfy the equation

$$2020^x + 2^y = 2024^z.$$

Serbia

Solution. Regarding the equation modulo 3, we obtain

$$1 + (-1)^y \equiv (-1)^z \pmod 3,$$

so we can conclude that y is even and z is odd. Let $y = 2y_1$, where y_1 is a positive integer. Since the largest powers of 2 in the factorisation of all three terms in the equation are respectively 2^{2x}, 2^{2y_1}, 2^{3z}, hence we rewrite the equation as

$$2^{2x} \cdot 505^x + 2^{2y_1} = 2^{3z} \cdot 253^z. \tag{1}$$

We will now determine which one of the integers $2x$, $2y_1$ is larger, in order to obtain the largest power of 2 dividing LHS.

Case 1: $2x > 2y_1$, i.e. $x > y_1$. Then, the largest power of 2 dividing LHS is 2^{2y_1}, while the largest power of 2 dividing RHS is 2^{3z}. Therefore, we derive $2y_1 = 3z$. Since z is odd, this is a contradiction.

Case 2: $2x < 2y_1$, i.e. $x < y_1$. Then, the largest power of 2 dividing LHS is 2^{2x}, while the largest power of 2 dividing RHS is 2^{3z}. Therefore, we derive $2x = 3z$. Since z is odd, this is a contradiction.

Thus, we must have $2x = 2y_1$, which implies $x = y_1$. The equation (1) then becomes

$$4^x \cdot (505^x + 1) = 2020^x + 4^x = 2024^z = 2^{3z} \cdot 253^z. \tag{2}$$

Since $2 \mid 505^x + 1$, but $4 \nmid 505^x + 1$, the largest exponent of 2 dividing $4^x(505^x + 1)$ is $2x + 1$, so we derive

$$2x + 1 = 3z. \tag{3}$$

Plugging it back to (2), it reduces to

$$505^x + 1 = 2 \cdot 253^z. \tag{4}$$

Observe that $x = 1$, $z = 1$ is a solution of the linear Diophantine equation (3), so all the solutions in positive integers are given by

$$x = 3k + 1, \quad z = 2k + 1, \quad k \in \mathbb{N}_0.$$

Now (4) becomes

$$505^{3k+1} + 1 = 2 \cdot 253^{2k+1}. \tag{5}$$

Obviously $k = 0$ is a solution of (5). If k is positive integer, then from the sequence of inequalites

$$505^{3k+1} + 1 > 505^{3k+1} = 505^{k+1} \cdot 505^{2k} > 506 \cdot 505^{2k} > 506 \cdot 253^{2k} = 2 \cdot 253^{2k+1}$$

which implies that there are no more solutions.

Hence, the only solution of (5) is $k = 0$, so $x = y_1 = 1$ and $z = 1$ and the only solution of the given equation is

$$(x, y, z) = (1, 2, 1).$$

4. Three friends Archie, Billie and Charlie play a game. At the beginning of the game, each of them has a pile of 2024 pebbles. Archie makes the first move, Billie makes the second, Charlie makes the third and they continue to make moves in the same order. In each move, the player making the move must choose a positive integer n greater than any previously chosen number by any player, take $2n$ pebbles from his pile and distribute them equally to the other two players. If a player cannot make a move, the game ends and that player loses the game.

   Determine all the players who have a strategy such that, regardless of how the other two players play, they will not lose the game.

North Macedonia

Solution. We will prove that only Charlie has a non-losing strategy. First we discuss what happens right before a player loses the game. Let t be the number chosen in the last move and let the losing player have s pebbles in his pile before the move. In order for the player to lose $2t + 2$ must be larger than $s + t$ so that he can't make the next move. This implies that $t \geq s - 1$, i.e. before the move the previous player must have at least $2s - 2$ pebbles. This means that if a player before his move have at least $2s - 2$ pebbles and the next player has s pebbles, then he can choose $s - 1$ no make the next player lose. This will leave the next player with $2s - 1$ pebble, disabling him to make a move. Also, if player X has s pebbles while the player Y has at most $2s - 3$ pebbles right before player Y plays, then player X guarantees not to lose the game in his turn.

Assume that at some point in the game the consecutive players have x, y, and z pebbles in their piles and they choose numbers u, $u + v$, and $u + v + w$ respectively. In these three moves we have:

$$
\begin{aligned}
(x, y, z) &\rightarrowtail (x - 2u, y + u, z + u) \rightarrowtail \\
(x - u + v, y - u - 2v, z + 2u + v) &\rightarrowtail \\
(x + 2v + w, y - v + w, z - v - 2w)&.
\end{aligned}
$$

Considering this the following two statements are true:

(i) If the player that plays second (after the position) plays $v = 1$, then after the three moves the number of pebbles in his pile does not decrease (since w is positive integer $y - 1 + w \geq y$).

(ii) If the player that plays third (after the position) plays $w = 1$, then after the three moves the number of pebbles in the second players pile does not increase (since v is positive integer $y - v + 1 \leq y$).

Now we will look at the cases for each player. Let a_i, b_i, c_i be the number Archie, Billie, Charlie chooses in their i-th turns, respectively.

Claim 1: Charlie has a non-losing strategy.

Proof: After the first move of Archie, Charlie has at least 2025 pebbles in his pile. If Charlie chooses $c_i = b_i + 1$, using (i) we conclude that after $3i + 1$ moves Charlie has at least 2025 pebbles in his pile since he's in the position of the middle player. In order for Charlie to lose, Billie must have at least $2 \cdot 2024$ pebbles at some point. However the total number of pebbles $3 \cdot 2024$ doesn't change throughout the game, hence this situation is impossible ($3 \cdot 2024 < 2 \cdot 2024 + 2025$). This means that Charlie has a non-losing strategy (choosing $c_i = b_i + 1$). □

Claim 2: Billie does not have a non-losing strategy.

Proof: Assume that Archie and Charlie play the strategies to choose $a_{i+1} = c_i + 1$ and $c_i = b_i + 1$ (where $c_0 = 0$ by definition). By (ii) the number of pebbles in Billie's pile after $3i$ moves does not increase. Since before the first move he has 2024 pebbles, he can't have more than 2024 pebbles at any point after $3i$ moves. If at some point (before his last move) Billie chooses $b_i > a_i + 1$ this increases the number of pebbles in Archie's pile before the $3i + 2$-nd move, to at least 2025, hence Archie cannot lose. Charlie is also playing a non-losing strategy, hence in this case Billie will lose. Otherwise, we have the situation:

$$(2024, 2024, 2024) \xrightarrow{1} (2022, 2025, 2025) \xrightarrow{2} (2024, 2021, 2027) \xrightarrow{3}$$
$$(2027, 2024, 2021) \xrightarrow{4} \cdots \xrightarrow{2019} (4043, 2024, 5) \xrightarrow{2020} (3, 4044, 2025).$$

After this Billie can choose 2021 or 2022, but both cases lead to him losing:

$$(3, 4044, 2025) \xrightarrow{2021} (2024, 2, 4046) \xrightarrow{2022} (4046, 2024, 2) \xrightarrow{2023} (0, 4047, 2025),$$
$$(3, 4044, 2025) \xrightarrow{2022} (2025, 0, 4047) \xrightarrow{2023} (4048, 2023, 1) \xrightarrow{2024} (0, 4047, 2025).$$

This means that Billie doesn't have a non-losing strategy. □

Claim 3: Archie does not have a non-losing strategy.

Proof: Let Billie and Charlie choose $b_i = a_i + 1$ and $c_i = b_i + 1$ until the last turn. If Archie chooses $a_i = c_{i-1} + 1$ in all his turns (while $c_0 = 0$), then again it leads to the case $(4043, 2024, 5)$. After this Archie can choose 2020 or 2021 and in both cases Billie and Charlie can make Archie lose as follows:

$$(4043, 2024, 5) \xrightarrow{2020} (3, 4044, 2025) \xrightarrow{2021} (2024, 2, 4046) \xrightarrow{2023} (4047, 2025, 0),$$
$$(4043, 2024, 5) \xrightarrow{2021} (1, 4045, 2026) \xrightarrow{2022} (2023, 1, 4048) \xrightarrow{2023} (4046, 2024, 2).$$

Now, assume that Archie chooses $a_i > c_{i-1} + 1$ before this case happens. Then let Billie choose $b_i = a_i + 1$ and Charlie choose $c_i = b_i + a_i - c_{i-1}$. In this case, before Archie moves, the players have $(2024+3k, 2024, 2024-3k)$ pebbles. If Archie chooses $\frac{2024+3k}{2} > L > 3k+1$, with the described moves, the number of pebbles at the end of this turn will be

$$(2024 + L + 2,\ 2024 + L - 3k - 1,\ 2024 - 2L + 3k - 1)$$

respectively for Archie, Billie and Charlie, and we see that all the moves are valid. Then Billie have $2024 + L - 3k - 1 \geq 2025$ pebbles when Archie moves, and Charlie have at least 2025 pebbles when Billie moves. Then, by (i), choosing $b_i = a_i + 1$ and $c_i = b_i + 1$ will guarantee that Billie and Charlie will not lose the game, hence in this situation Archie loses.

If Archie chooses $\frac{2024+3k}{2} = L < 2023$, then, after he plays we have the case $(0, 2024 + L, 4048 - L)$. Let Billie choose $L + 1$, which he can, then we have the case $(L + 1, 2022 - L, 4049)$. Here, Charlie can choose 2024 and Archie has no move, so he loses.

Hence, Archie does not have a non-losing strategy. □

Finally, we conclude that only Charlie has a non-losing strategy.