Language: English

Thursday, June 27, 2024

Problem 1. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 = \frac{1}{4}.$$

Prove that

$$\frac{1}{\sqrt{b^2+c^2}} + \frac{1}{\sqrt{c^2+a^2}} + \frac{1}{\sqrt{a^2+b^2}} \le \frac{\sqrt{2}}{(a+b)(b+c)(c+a)}.$$

Problem 2. Let ABC be a triangle such that $AB < AC$. Let the excircle opposite to A be tangent to the lines AB, AC and BC at points D, E and F, respectively, and let J be its centre. Let P be a point on the side BC. The circumcircles of the triangles BDP and CEP intersect for the second time at Q. Let R be the foot of the perpendicular from A to the line FJ. Prove that the points P, Q and R are collinear.

(The *excircle* of a triangle ABC opposite to A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 3. Find all triples of positive integers (x, y, z) that satisfy the equation

$$2020^x + 2^y = 2024^z.$$

Problem 4. Three friends Archie, Billie and Charlie play a game. At the beginning of the game, each of them has a pile of 2024 pebbles. Archie makes the first move, Billie makes the second, Charlie makes the third and they continue to make moves in the same order. In each move, the player making the move must choose a positive integer n greater than any previously chosen number by any player, take $2n$ pebbles from his pile and distribute them equally to the other two players. If a player cannot make a move, the game ends and that player loses the game.

Determine all the players who have a strategy such that, regardless of how the other two players play, they will not lose the game.

Time is 4 hours and 30 minutes
Each problem is worth 10 points